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Note on a heterogeneous shear flow

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Goldstein (1931) has considered the stability of a shear layer within which the velocity and the density vary linearly and outside which they are constant. Rayleigh (1880, 1887) had found that the corresponding, homogeneous shear flow is unstable in and only in a finite band of wave-numbers. Goldstein concluded that a small density gradient renders the flow unstable for all wave-numbers. This conclusion appears to depend on the acceptance of all possible branches of a multi-valued eigenvalue equation, and it is shown that the principal branch of this eigenvalue equation yields one and only one unstable mode if and only if the wave-number lies in a band that decreases from Rayleigh's band to zero as the Richardson number increases from 0 to $\frac{1}{4}$.

1. Introduction

Let hx and hy be Cartesian co-ordinates referred to a characteristic length h, with the y-axis directed vertically, let V be a characteristic velocity, and let

$$\langle y\rangle = \left\{\begin{array}{lll} 1 & \text{for} & y \geqslant 1 \\ y & \text{for} & -1 \leqslant y \leqslant 1 \\ -1 & \text{for} & y \leqslant -1 \end{array}\right\}. \tag{1.1}$$

Following Goldstein (1931), we consider the stability of the two-dimensional, heterogeneous shear flow described by the velocity profile

$$U(y) = V\langle y\rangle, \tag{1.2}$$

and the density profile

$$\rho(y) = \rho_0[1-\sigma\langle y\rangle] \quad (0<\sigma\ll 1) \tag{1.3}$$

in a perfect, incompressible fluid. The Richardson number for the shear layer ($|y|<1$) is given by

$$J = \sigma g h/V^2. \tag{1.4}$$

The restriction $\sigma \ll 1$ permits the usual Boussinesq approximation, by virtue of which the parameter σ enters the stability problem only through the parameter J.

Assuming a small displacement

$$\eta(x,y,t) = \mathrm{Re}\{F(y)\,e^{i\alpha(x-ct)}\} \quad (\alpha>0,\ c=c_r+ic_i) \tag{1.5}$$

of the streamlines from their mean positions, where α is a dimensionless wave number and c is a dimensionless wave-speed, we seek the neutral curve that bounds the domain of unstable disturbances ($c_i > 0$) in an (α, J)-plane. We designate the disturbances comprised by this neutral curve as *singular neutral modes*.

Rayleigh (1880) considered the homogeneous ($\sigma = J = 0$) shear flow described by (1.2) and found that: singular neutral modes exist for $\alpha = 0$ and $\alpha = \alpha_1 = 0{\cdot}639$; these modes are stationary ($c = 0$); and $c^2 < 0$ for $0 < \alpha < \alpha_1$, so that the principle of exchange of stabilities holds. Goldstein (1931) considered the heterogeneous shear flow described by (1.2) and (1.3) and concluded that 'a slight heterogeneity ($0 < J \ll 1$) causes instability for all wavelengths'. This rather unexpected conclusion (see remarks in last paragraph of § 1, Miles 1961) does not appear to have been refuted in the literature.

It appears that Goldstein's conclusion depends on the acceptance of all possible branches of a multi-valued eigenvalue equation,† say

$$\Delta(c; \alpha, \nu) = 0, \tag{1.6}$$

where

$$\nu = (\tfrac{1}{4} - J)^{\frac{1}{2}}. \tag{1.7}$$

We shall accept only a single branch of Δ and shall show that: the neutral curve $J = J_0(\alpha)$ is unique and single-valued, rising from $(\alpha, J) = (0, 0)$ to $(\alpha_m, \frac{1}{4})$ and then descending to $(\alpha_1, 0)$; one and only one singular neutral mode exists for each (α, J)-point on this curve; and this mode is stationary. It then follows from a general consideration of antisymmetric shear flows (Miles 1963) that there exists one and only one unstable disturbance for each (α, J)-point under the neutral curve $[J < J_0(\alpha)]$ and that this mode experiences a simple exponential instability.

Briefly stated, then, density stratification modifies Rayleigh's result by reducing the α-band of instability from $(0, \alpha_1)$ to $(\alpha_m -, \alpha_m +)$ as J increases from 0 to $\frac{1}{4}$.

2. Eigenvalue equation

Goldstein's solution for $F(y)$ (actually he worked with the dependent variable $(U-c)F$) in the shear layer yields

$$F(y) = z^{-\frac{1}{2}}[AI_\nu(z) + BI_{-\nu}(z)], \tag{2.1}$$

where

$$z = \alpha(y - c), \tag{2.2}$$

I_ν is a modified Bessel function, ν is given by (1.7), and A and B must satisfy the homogeneous equations implied by

$$F'(y) \pm \alpha F(y) = 0 \quad \text{for} \quad y = \pm 1. \tag{2.3}$$

Assuming $c_i > 0$ ($c_i \to 0+$ for a singular neutral mode) and requiring $F(y)$ to be continuous in $y = (-1, 1)$, we can restrict the argument of z according to

$$-\pi < \arg z < 0 \quad (c_i > 0) \tag{2.4}$$

and continue the individual solutions around the branch point at $z = 0$ according to

$$z^{-\frac{1}{2}} I_{\pm\nu}(z) = e^{(\frac{1}{2} \mp \nu) i\pi}(-z)^{-\frac{1}{2}} I_{\pm\nu}(-z). \tag{2.5}$$

† Prof. Goldstein (verbal communication) agrees with this assertion.

Introducing

$$f(z,\nu) = z^{-\nu}[zI'_\nu(z) + (z-\tfrac{1}{2})\,I_\nu(z)] \tag{2.6}$$

and

$$Z_\pm = \alpha(1\mp c) \tag{2.7}$$

and invoking (2.4) and (2.5), we can place the eigenvalue equation implied by (2.1)–(2.3) in the form (1.6), with

$$\Delta(c,\alpha,\nu) = \begin{vmatrix} Z_+^\nu f(Z_+,\nu) & z_+^{-\nu} f(Z_+,-\nu) \\ Z_-^\nu f(Z_-,\nu)\,e^{-i\nu\pi} & Z_-^{-\nu} f(Z_-,-\nu)\,e^{i\nu\pi} \end{vmatrix} \tag{2.8a}$$

$$= e^{i\nu\pi}(Z_+/Z_-)^\nu f(Z_+,\nu) f(Z_-,-\nu) - e^{-i\nu\pi}(Z_+/Z_-)^{-\nu} f(Z_+,-\nu) f(Z_-,\nu) \tag{2.8b}$$

and

$$0 > \arg(Z_+/Z_-)^\nu > -\nu\pi > -\tfrac{1}{2}\pi \quad (c_i > 0, 0 < \nu < \tfrac{1}{2}). \tag{2.9}$$

It is only in this last restriction that our discussion differs from that given by Goldstein; otherwise, our eigenvalue equation is equivalent to his (5.15). We observe that Δ is an odd function of ν; accordingly, we need determine c only for ν in the positive range $(0, \frac{1}{2})$ if $0 < J < \frac{1}{4}$. We also observe that $f(z,\nu)$ is an entire function of each of z and ν.

Let us consider, for example, the possible zeros of Δ as $\alpha \to 0$. Letting $Z_\pm \to 0$ in (2.8*b*), we obtain

$$\Delta = (\tfrac{1}{4}-\nu^2)(\pi\nu)^{-1}\sin(\pi\nu)\,[e^{i\nu\pi}(Z_+/Z_-)^\nu - e^{-i\nu\pi}(Z_+/Z_-)^{-\nu}]\,[1+O(Z_\pm)] \quad (Z_\pm \to 0). \tag{2.10}$$

This has no zeros under the restriction (2.9), but it yields (cf. Goldstein's (5.72))

$$c = \pm i\cot(r\pi/2\nu) \quad (r=1,2,\ldots), \tag{2.11}$$

if we accept all branches of Δ, qua function of c with branch points at, but no branch cuts from, $c = \pm 1$. We observe that at least some of the zeros given by (2.11) for any real value of ν lie outside the circle $|c| = 1$, whereas unstable modes associated with the velocity profile (1.2) must lie within this circle (Howard 1961).

3. Singular-neutral mode

We can establish (Miles 1961) that necessary conditions for the existence of a singular neutral mode are:† $-1 < c < 1$; $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$; and $F(y)$ must be of one exponent $(-\frac{1}{2}+\nu$ or $-\frac{1}{2}-\nu)$, rather than a linear combination of the solutions of both exponents, in the neighbourhood of the singular point $y = c$. It follows that either $A = 0$ or $B = 0$ in (2.1), and (2.3) then implies that c must satisfy the simultaneous equations

$$f[\alpha(1\mp c),\nu] = 0 \quad (-1<c<1, -\tfrac{1}{2}<\nu<\tfrac{1}{2}). \tag{3.1}$$

We shall prove that $f(z,\nu)$ has one and only one positive-real zero for $-\frac{1}{2} < \nu < \frac{1}{2}$; accordingly, (3.1) can be satisfied only for $c = 0$ and implies the neutral curve

$$f(\alpha,\nu) = 0 \quad (c=0, \alpha>0, -\tfrac{1}{2}<\nu<\tfrac{1}{2}), \tag{3.2}$$

on which α is a single-valued function of ν. (We note that the number of zeros of the entire function $f(z,\nu)$ in a sufficiently large circle in a complex-z plane is equal to the number of zeros of $z^{1-\nu}I_\nu(z)$ in that circle by virtue of Rouché's theorem.)

† The proofs given by Miles (1961) were for the boundary conditions $F = 0$ at $y = y_1, y_2$, but extensions for the boundary conditions (2.3) are straightforward.

Let
$$H_\nu(z) = zI'_\nu(z)/I_\nu(z) = \tfrac{1}{2} - z + [z^\nu f(z,\nu)/I_\nu(z)], \tag{3.3}$$
under which transformation the differential equation for $I_\nu(z)$ goes over to the Riccati equation
$$zH'_\nu(z) + H^2_\nu(z) = \nu^2 + z^2. \tag{3.4}$$

Making use of the known, positive-definite integral
$$2\int_0^z I^2_\nu(z)\, z\, dz = (\nu^2+z^2) I^2_\nu(z) - z^2 I'^2_\nu(z)$$
$$= I^2_\nu(z)\,[\nu^2+z^2-H^2_\nu(z)], \tag{3.5}$$
we deduce that
$$zH'_\nu(z) > 0. \tag{3.6}$$

It follows that $H_\nu(z)$ increases monotonically from $H_\nu(0) = \nu$ to its asymptotic value of $+(\nu^2+z^2)^{\frac{1}{2}}$. Invoking the restriction $-\frac{1}{2} < \nu < \frac{1}{2}$, we infer that $H_\nu(z) = \frac{1}{2} - z$ has one and only one positive-real root. Invoking the known fact that $z^{-\nu}I_\nu(z)$ has no real zeros, we conclude that $f(z,\nu)$ has one and only one positive-real zero.

ν	α	J
$\frac{1}{2}$	0	0
0	α_m	$\frac{1}{4}$
$-\frac{1}{2}$	α_t	0

TABLE 1

Rather more can be said if we restrict ν to be positive, for then
$$\nu^2 < H^2_\nu(z) < \nu^2+z^2. \tag{3.7}$$

Integrating $H'_\nu(z)$, as given by (3.4), between
$$(z = 0, H_\nu = \nu) \quad \text{and} \quad (z = \alpha, H_\nu = \tfrac{1}{2} - \alpha),$$
we obtain
$$\nu = \tfrac{1}{2} - \alpha - \tfrac{1}{2}\alpha^2 + \int_0^\alpha [H^2_\nu(z) - \nu^2]\, z^{-1}\, dz. \tag{3.8}$$

Bounding this last integral with the aid of (3.7), we obtain
$$\tfrac{1}{2} - \alpha - \tfrac{1}{2}\alpha^2 < \nu < \tfrac{1}{2} - \alpha \quad (\nu > 0). \tag{3.9}$$

We can characterize the neutral curve in an (α, J)-plane by the table 1, where $\alpha = 0$, α_1 are the Rayleigh end-points and $\alpha = \alpha_m$ locates the maximum. We already know that $\alpha_1 \simeq 0{\cdot}639$, and we deduce from (3.9) that $\sqrt{2} - 1 < \alpha_m < \frac{1}{2}$; a direct calculation from (3.2) yields $\alpha_m = 0{\cdot}415$. The complete curve is plotted in figure 1.

We have also computed the growth rates in the unstable range. Though the eigenvalue relation $\Delta = 0$ with the restriction (2.9) is expressed in terms of the modified Bessel functions, this computation was actually done by a direct numerical integration of the Riccati equation associated with the linear second-order stability equation. For each of a series of values of Richardson number J and wave-number α, the equation was integrated (using a Runge–Kutta method)

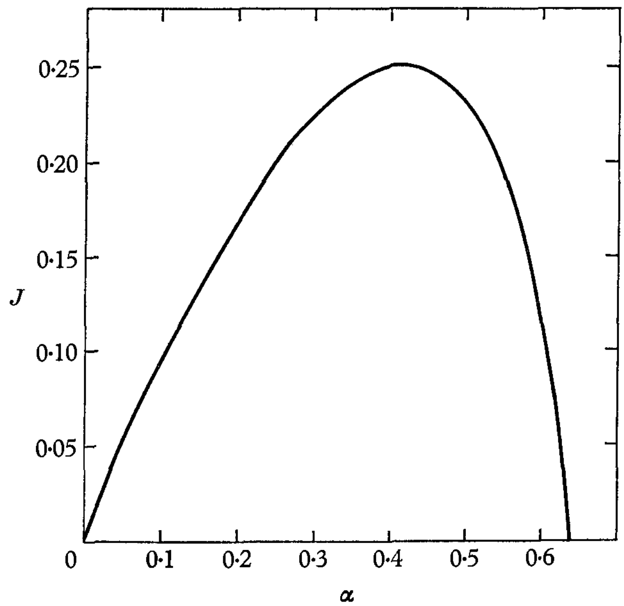

FIGURE 1

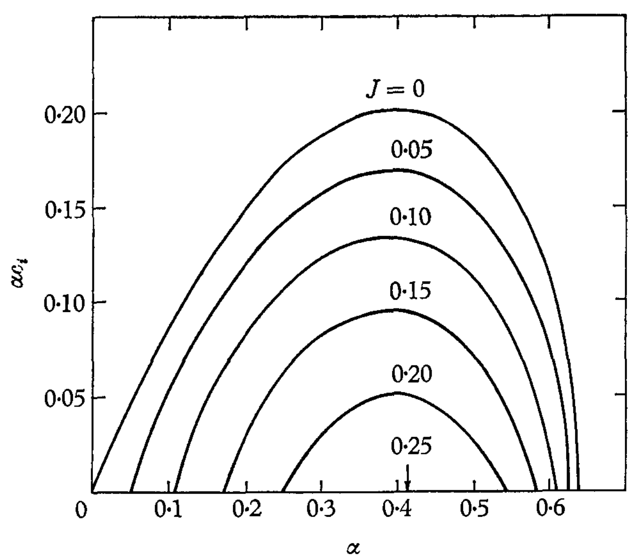

FIGURE 2

from one end, the value of c_i being adjusted to satisfy the appropriate condition at the other. The results are shown in figure 2, which gives the growth rate αc_i as a function of α for various values of J.

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